we have

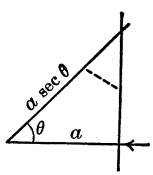
$$(\cos x + i \sin x)(\cos y + i \sin y) = 1/Z = \overline{Z} = \cos(-x-y) - i \sin(-x-y)$$
$$= \cos(x+y) + i \sin(x+y),$$

which is De Moivre's formula. From this the trigonometric addition or subtraction formulas are easily derived in all their generality.

SAILING TO WINDWARD.

BY W. E. BYERLY, Harvard University.

The problem of sailing to a windward goal is usually the simple problem of *getting to windward.* For, if we find the straight path of least time from the boat to a perpendicular to the wind's direction, by tacking at the proper point we shall reach any goal in that perpendicular in the time it would take to reach the perpendicular had we continued on the original course.



The velocity of the boat on any course is some function of θ depending on the model and sailing qualities of the boat, where θ is the angle which the course makes with the direction of the wind, and we shall suppose this function given, and shall represent it by $F(\theta)$.

The time required to get the distance a to windward is the distance sailed, $a \sec \theta$, divided by the velocity, $F(\theta)$, and we wish to make this time a minimum.

Let $u = a \sec \theta / F(\theta)$. Then

$$\frac{du}{d\theta} = \frac{F(\theta)\, a \sec \theta \tan \theta - a \sec \theta F'(\theta)}{[F(\theta)]^2} = 0, \qquad \text{where} \qquad F'(\theta) = \frac{d}{d\theta} F(\theta),$$

or

$$F(\theta) \tan \theta - F'(\theta) = 0. \tag{1}$$

The solution of this equation will give us the angle which our course on each tack should make with the direction of the wind, whether we are using sails alone or sails and auxiliary motor.

In any concrete case equation (1) can be solved by "trial and error" with the aid of a three-place trigonometric table (preferably giving the angles in radians as well as in degrees) with sufficient accuracy for all practical purposes.

The function $F(\theta)$ ought to be determined by experiment for every boat, but as a first approximation, in the case of a sail boat, it may be taken as equal to $k(\theta - \alpha)$, where α, "the angle of repose," is the angle with the direction of the wind within which the boat will not sail; an angle, by the way, that is very easily discovered by experimenting with the boat.

Equation (1) becomes $k(\theta - \alpha) \tan \theta - k = 0$ or

$$\theta - \alpha = \operatorname{ctn} \theta. \tag{2}$$

The θ found from this equation is the angular distance off the wind which we should sail in going to windward, and is evidently independent of the wind's velocity.

For a rather sluggish boat, for which α is 30°, θ is about 60°.

Let us now suppose that the sail boat has a gasolene engine which can give her a velocity v. Then with sail and engine, $F(\theta) = v + k(\theta - \alpha)$, and equation (1) becomes $[v + k(\theta - \alpha)] \tan \theta - k = 0$ or

$$\frac{v}{k} + (\theta - \alpha) = \operatorname{ctn} \theta. \tag{3}$$

Here θ is no longer the same for all breezes but k is easily determined by shutting off the motor and noting the speed on any convenient course, remembering that this speed must be $k(\theta - \alpha)$.

If $\alpha = 30°$, $v = 6$, and $k = 8$, equation (3) gives about 45° for θ.

We shall get to windward, then, under sail and gas combined, most rapidly if we keep the boat about four points off the wind. Can we perhaps do better by dropping sail and going dead to windward under gas alone? To get a mile to windward under sail and motor will take us $(\sec \theta)/[v + k(\theta - \alpha)]$ hours. Computing this value for $\theta = 45°$, $\alpha = 30°$, $v = 6$, and $k = 8$, we get 0.175 hour. To get a mile to windward under gas takes 0.167 hour.

The time required to get a mile to windward under gas on a course making the angle β with the wind is $(\sec \beta)/v$. This is less than 0.175 if β is less than 18°.

Summing up, we see that if short-handed our "sailing directions" should read as follows:

(*a*) For any goal lying in the angle whose vertex is at the boat and whose sides are inclined 18° to the wind's direction, drop sail and head for the goal under gas alone.

(*b*) For any goal between these lines and lines through the boat and inclined 45° to the wind's direction, tack under sail and gas, keeping four points off the wind on each tack.

(*c*) For all other goals, go straight under sail and gas.

For a second and much closer approximation, let us assume that the velocity under sail is $a(\theta - \alpha) + b(\theta - \alpha)^2$. In practice a and b can be determined from the observed speed on two courses, preferably chosen so that the angle $\theta - \alpha$ is small.

For sail alone $F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2$, and equation (1) becomes $[a(\theta - \alpha) + b(\theta - \alpha)^2] \tan \theta = a + 2b(\theta - \alpha)$ or

$$\left(\tan \theta - \frac{2b}{a}\right)(\theta - \alpha) + \frac{b}{a}(\theta - \alpha)^2 \tan \theta = 1. \tag{4}$$

For an example, suppose as before that $\alpha = 30°$, and suppose that the boat it found to make 2 knots when $\theta - \alpha$ is $\frac{1}{4}$ and 3 knots when $\theta - \alpha$ is $\frac{1}{2}$.

We have

$$2 = \frac{a}{4} + \frac{b}{16}; \qquad 3 = \frac{a}{2} + \frac{b}{4}.$$

Whence $a = 10, b = -8, b/a = -0.8$. Equation (4) gives $(\tan\theta + 1.6)(\theta - \alpha) - 0.8(\theta - \alpha)^2 \tan\theta = 1$, and

$$\theta = 55° \text{ approximately.}$$

If we use sail and motor, $F(\theta) = v + a(\theta - \alpha) + b(\theta - \alpha)^2$. Equation (3) becomes

$$[v + a(\theta - \alpha) + b(\theta - \alpha)^2] \tan\theta = a + 2b(\theta - \alpha)$$

or

(5) $$\left(\tan\theta - \frac{2b}{a}\right)(\theta - \alpha) + \left[\frac{b}{a}(\theta - \alpha)^2 + \frac{v}{a}\right]\tan\theta = 1.$$

For our example, equation (5) becomes

$$(\tan\theta + 1.6)(\theta - \alpha) + [0.6 - 0.8(\theta - \alpha)^2] \tan\theta = 1,$$

$$\theta = 41°.5 \text{ approximately.}$$

The best time for a mile to windward under sail and motor is

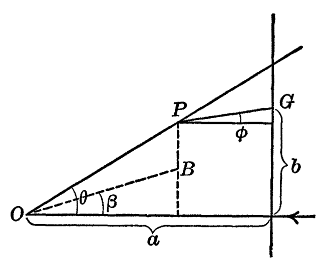
$$(\sec\theta)/[v + a(\theta - \alpha) + b(\theta - \alpha)^2],$$

or about 0.1738 hour. Under motor alone, in this time, the boat would go 1.043 miles, and $\sec^{-1} 1.043$ is about 16°.5.

Our sailing directions would be the same as in our first example, except that in (*a*) 18° would be changed to 16°.5, and in (*b*) 45° would be changed to 41°.5.

Hitherto, we have supposed the crew short-handed, so that sail was neither raised nor lowered in transit.

In a light wind where β is an angle of appreciable magnitude and we are aiming at a goal in the sector $\theta - \beta$, we can do best by using sail and motor



part way and motor only the remainder of the way. Let us consider this new problem. Let θ and β have their old signification and let φ be the angle which

the course makes with the wind after we lower sail. Let P be the turning point and G be the goal. Then, since the whole time of transit is the time that it would take to go the distance $OB + PG$ under motor, we must choose φ so that this distance shall be a minimum.

$$PG = \frac{(a \tan \theta - b) \cos \theta}{\sin (\theta - \varphi)}, \qquad OB = (a - PG \cos \varphi) \sec \beta.$$

$$OB + PG = a \sec \beta - PG(\cos \varphi \sec \beta - 1).$$

This will be a minimum when $PG \cos \varphi(\sec \beta - \sec \varphi)$ is a maximum; and that will be the case when u is a maximum if

$$u = \frac{\cos \theta \cos \varphi}{\sin (\theta - \varphi)} [\sec \beta - \sec \varphi] = \frac{\sec \beta - \sec \varphi}{\tan \theta - \tan \varphi}.$$

$$\frac{du}{d\varphi} = \frac{(\tan \varphi - \tan \theta) \sec \varphi \tan \varphi + (\sec \beta - \sec \varphi) \sec^2 \varphi}{(\tan \theta - \tan \varphi)^2} = 0.$$

$$(\tan \theta - \tan \varphi) \tan \varphi = (\sec \beta - \sec \varphi) \sec \varphi.$$

$$\tan \theta \tan \varphi + 1 = \sec \beta \sec \varphi.$$

$$\cos \varphi + \tan \theta \sin \varphi = \sec \beta. \tag{6}$$

φ determined from equation (6) gives the bearing of the goal when the proper turning point is reached.

If $\theta = 45°$ and $\beta = 18°$, $\varphi = 3°$. If $\theta = 41°.5$ and $\beta = 16°.5$, $\varphi = 3°-$.

If the yachtsman has patience and a soul that longs for accuracy he can amuse himself by trying a third approximation for $F(\theta)$, namely,

$$F(\theta) = a(\theta - \alpha) + b(\theta - \alpha)^2 + c(\theta - \alpha)^3.$$

THE ACCELERATIONS OF THE POINTS OF A RIGID BODY.

By PETER FIELD and ALEXANDER ZIWET, University of Michigan.

Introduction.

It has long been known that, in plane motion, the acceleration field is completely determined by the accelerations of any two points; and the point of zero acceleration, or "acceleration center," for such a motion is discussed in most works on mechanics.[1] The corresponding problems for three dimensions, that is, the determination of the acceleration field from the accelerations of three points and the construction of the acceleration center (or centers), are less widely known. Vector methods are particularly appropriate for their solution. They

[1] Compare American Mathematical Monthly, Vol. XXI, 1914, pp. 105–113.